Scaling laws of formal reasoning

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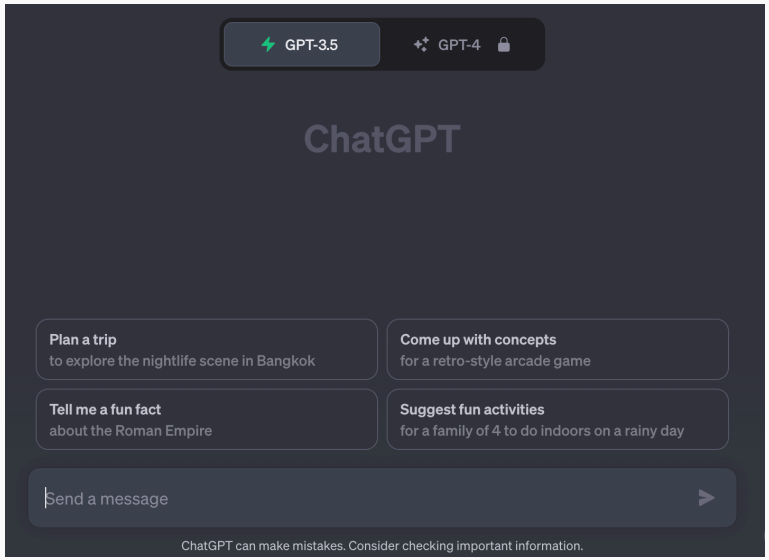
Foundation models and large language models (LLMs)

Platform for sequence generation

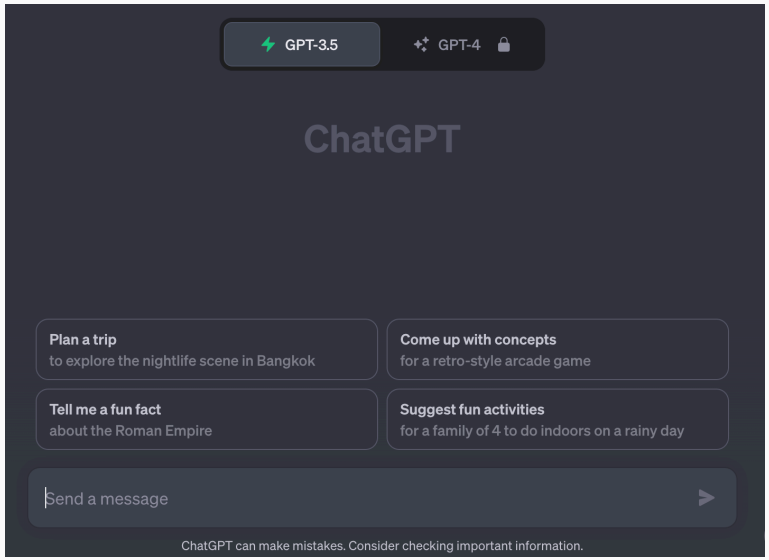
- Summarize documents
- Generate code from a screenshot
- ...



Foundation models and large language models (LLMs)

Core idea:

- Collect diverse data
- Train generative model
- Adapt to tasks (e.g., prompting)



Language models

Language models in an expert domain

- Finance
- Medicine
- Mathematics

Language models

Language models in an expert domain

- Finance
- Medicine
- Mathematics
  - Help generate verified proofs
  - Solve problems
  - Open-ended interactions
  - ...

[1] Many CMSA New Technologies talks, e.g.: 1/2021 (Christian Szegedy), 3/2022 (Stan Polu), 10/2022 (Guy Gur-Ari), 10/2023 (Alex Gu), 12/2023 (Katherine Collins), ...

Problem solving

Input:
Let $f(r) = \sum_{j=2}^{2008} \frac{1}{j^r} = \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{2008^r}$. Find $\sum_{k=2}^{\infty} f(k)$.

LLEMMA 34B solution:
We have

$$\begin{aligned}
\sum_{k=2}^{\infty} f(k) &= \sum_{k=2}^{\infty} \sum_{j=2}^{2008} \frac{1}{j^k} = \sum_{j=2}^{2008} \sum_{k=2}^{\infty} \frac{1}{j^k} = \sum_{j=2}^{2008} \frac{1}{j^2} \sum_{k=0}^{\infty} \frac{1}{j^k} = \sum_{j=2}^{2008} \frac{1}{j^2} \frac{1}{1-\frac{1}{j}} \\
&= \sum_{j=2}^{2008} \frac{1}{j(j-1)} = \sum_{j=2}^{2008} \left(\frac{1}{j-1} - \frac{1}{j} \right) \\
&= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{2007} - \frac{1}{2008} \right) \\
&= 1 - \frac{1}{2008} \\
&= \boxed{\frac{2007}{2008}}.
\end{aligned}$$

Final Answer: The final answer is $\frac{2007}{2008}$.

Figure 1: A LLEMMA 34B solution to a MATH [1] problem.

Formal theorem proving

Problem (MATH Number theory 185): When a number is divided by 5, the remainder is 3. What is the remainder when twice the number is divided by 5? Show that it is 1.

Human-written informal proof: If our number is n, then $n \equiv 3 \pmod 5$. This tells us that

$$2n = n + n \equiv 3 + 3 \equiv 1 \pmod 5.$$

The remainder is 1 when the number is divided by 5.

Informal-to-formal (Isabelle):
{Problem, human-written informal proof}

```
theorem mathd_numbertheory_185:
  fixes n ::nat
  assumes "n mod 5 = 3"
  shows "(2 * n) mod 5 = 1"

proof -
  have "2 * n = n + n"  <ATP>
  also have "... mod 5 =
    (n mod 5 + n mod 5) mod 5" <ATP>
  also have "... = (3 + 3) mod 5"
    using assms <ATP>
  also have "... = 1" <ATP>
  finally show ?thesis <ATP>
qed
```

Formal-to-formal (Lean 4):

```
theorem mathd_numbertheory_185
  (n : ℕ) (h₀ : n % 5 = 3)
  : 2 * n % 5 = 1 := by

-- INPUT (step 1):
--   n: ℕ
--   h₀: n % 5 = 3
--   ⊢ 2 * n % 5 = 1
rw [mul_mod, h₀]

-- INPUT (step 2):
--   n: ℕ
--   h₀: n % 5 = 3
--   ⊢ 2 % 5 * 3 % 5 = 1
simp only [h₀, mul_one]
```

Scaling laws

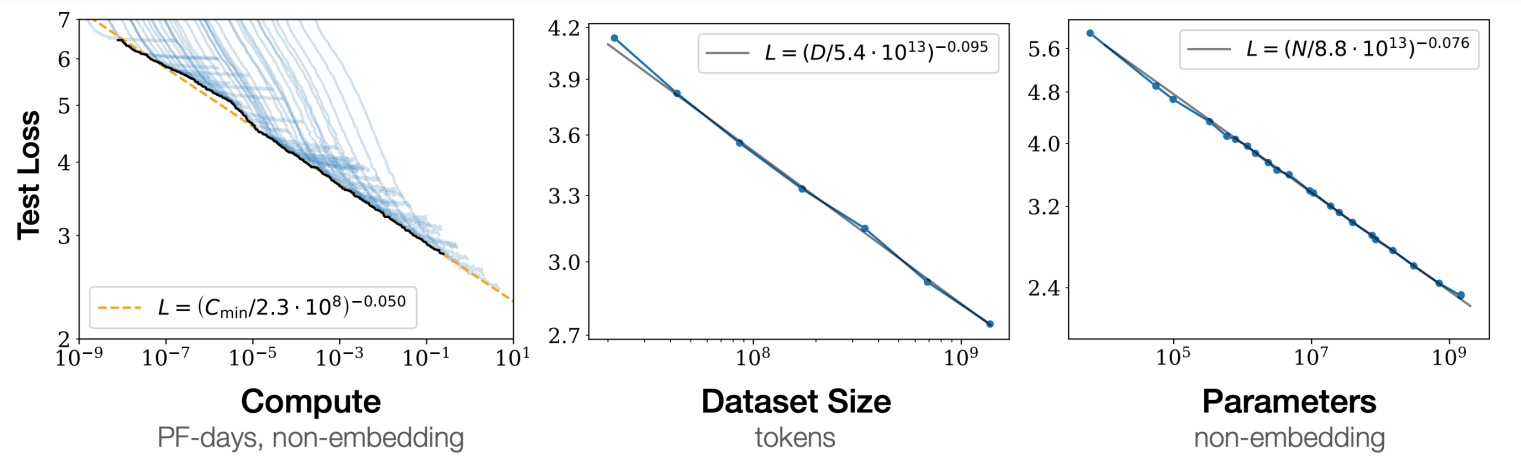


Figure 2: Increasing compute predictably improves language modeling.[1]

How do we predictably improve mathematical reasoning?

[1] Image from [Kaplan et al 2020]. See [2, 4] for more recent scaling laws.

Today's talk

1. Scaling data *quality* and *quantity*
    - $\rightarrow$ Llemma and ProofPile II
2. Scaling *inference* and *evaluation*
    - $\rightarrow$ Easy-to-hard evaluation

I. Llemma

Llemma: An Open Language Model For Mathematics

Zhangir Azerbayev, Hailey Schoelkopf, Keiran Paster, Marco Dos Santos, Stephen McAleer, Albert Q. Jiang, Jia Deng, Stella Biderman, Sean Welleck

ICLR 2024.

Approach 1: train a good generalist

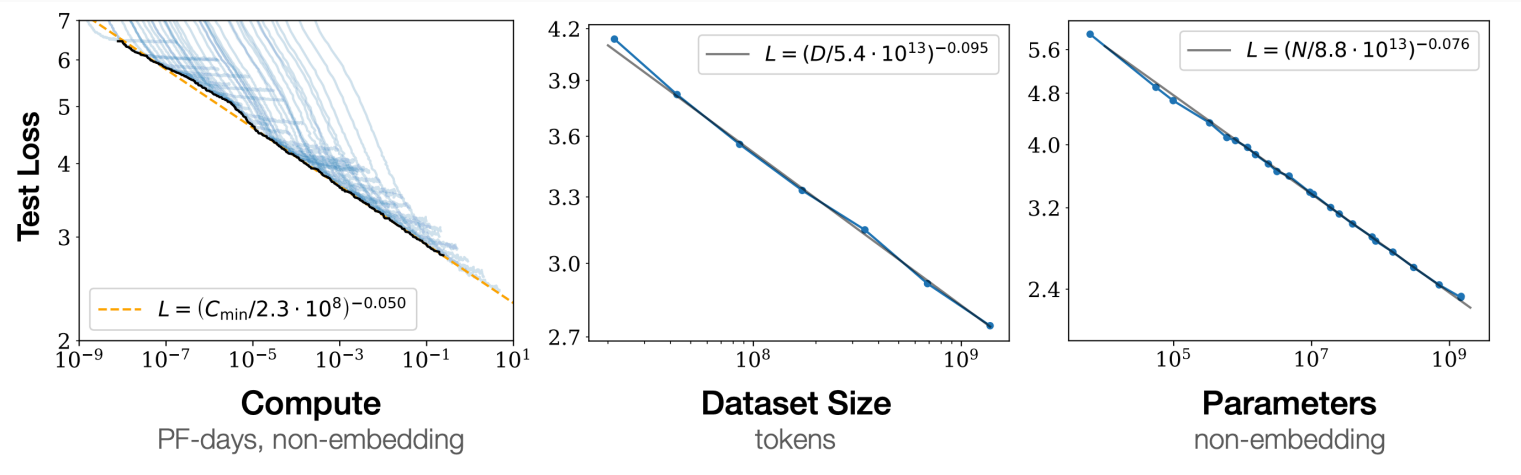


Figure 3: Increasing compute predictably improves language modeling.[2]

[2] Image from [Kaplan et al 2020]. See [2, 4] for more recent scaling laws.

Approach 1: train a good generalist

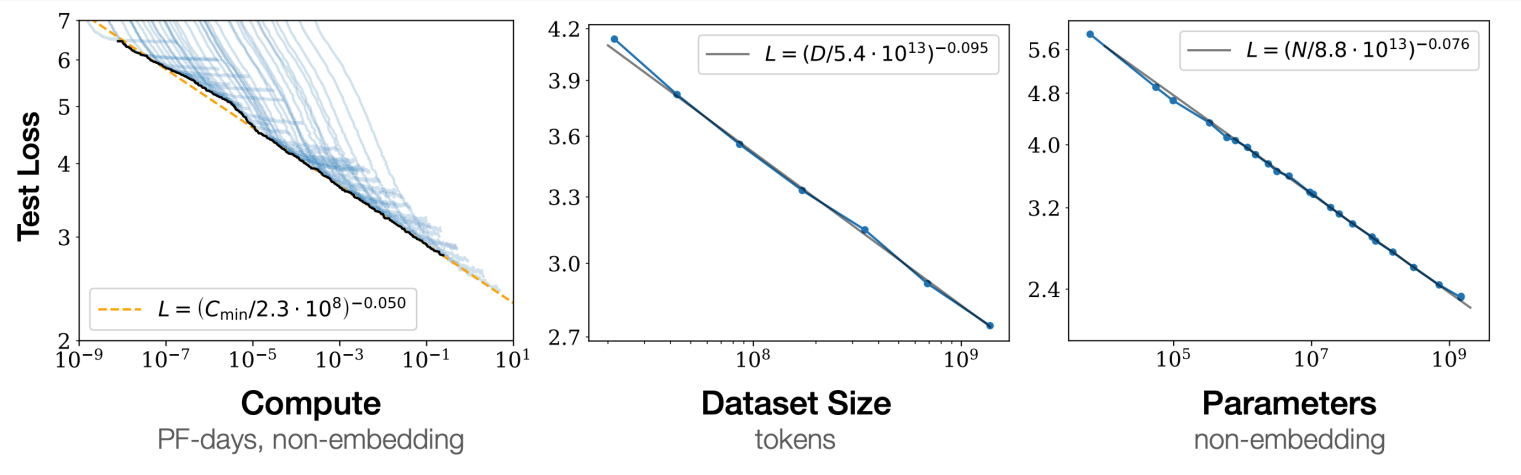


Figure 3: Increasing compute predictably improves language modeling.[2]

- *Idea:* let $\mathcal{D}$ be as general as possible (with math as a subset), increase compute as much as possible ($|\mathcal{D}|$ and $|\theta|$).

[2] Image from [Kaplan et al 2020]. See [2, 4] for more recent scaling laws.

Train a good generalist?

Example: Llama 2

- θ : 7B parameter transformer
- $\mathcal{D}$: 2 trillion tokens
- *General domain*: CommonCrawl, Github, Wikipedia, Arxiv, ...

Train a good generalist?

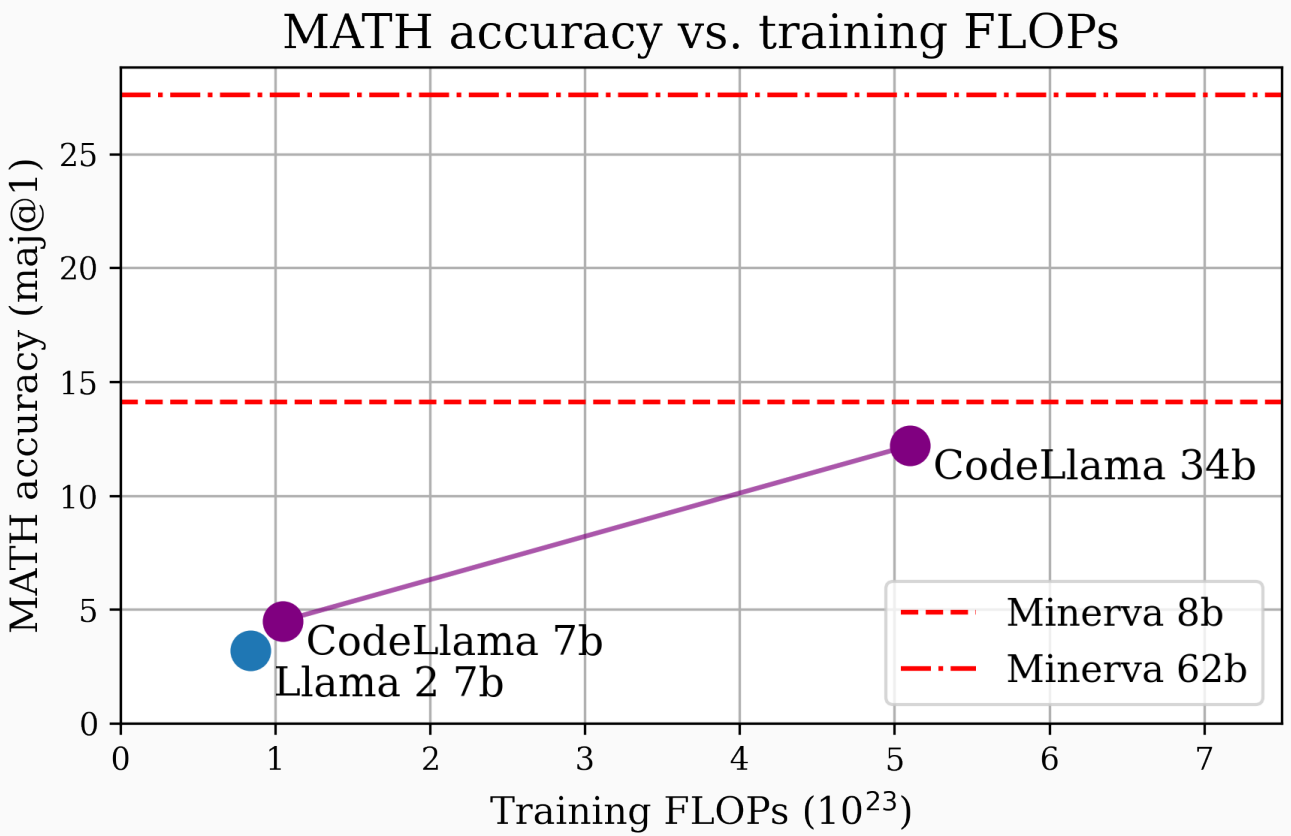


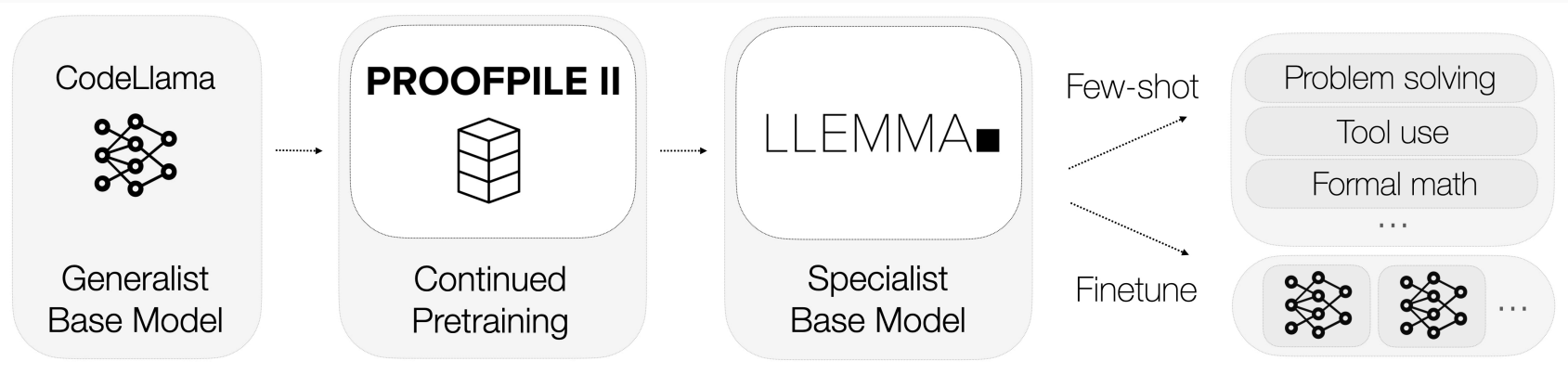
Figure 4: Training a good generalist can be inefficient

Minerva: Google LLM finetuned on math webpages + papers

Approach 2: specialize a generalist model

Recipe for specializing a foundation model to mathematical data:

- Collect high-quality, diverse math-related corpus, PROOFPILE II
- Continue pretraining a base model (e.g., Code LLama) on PROOFPILE II



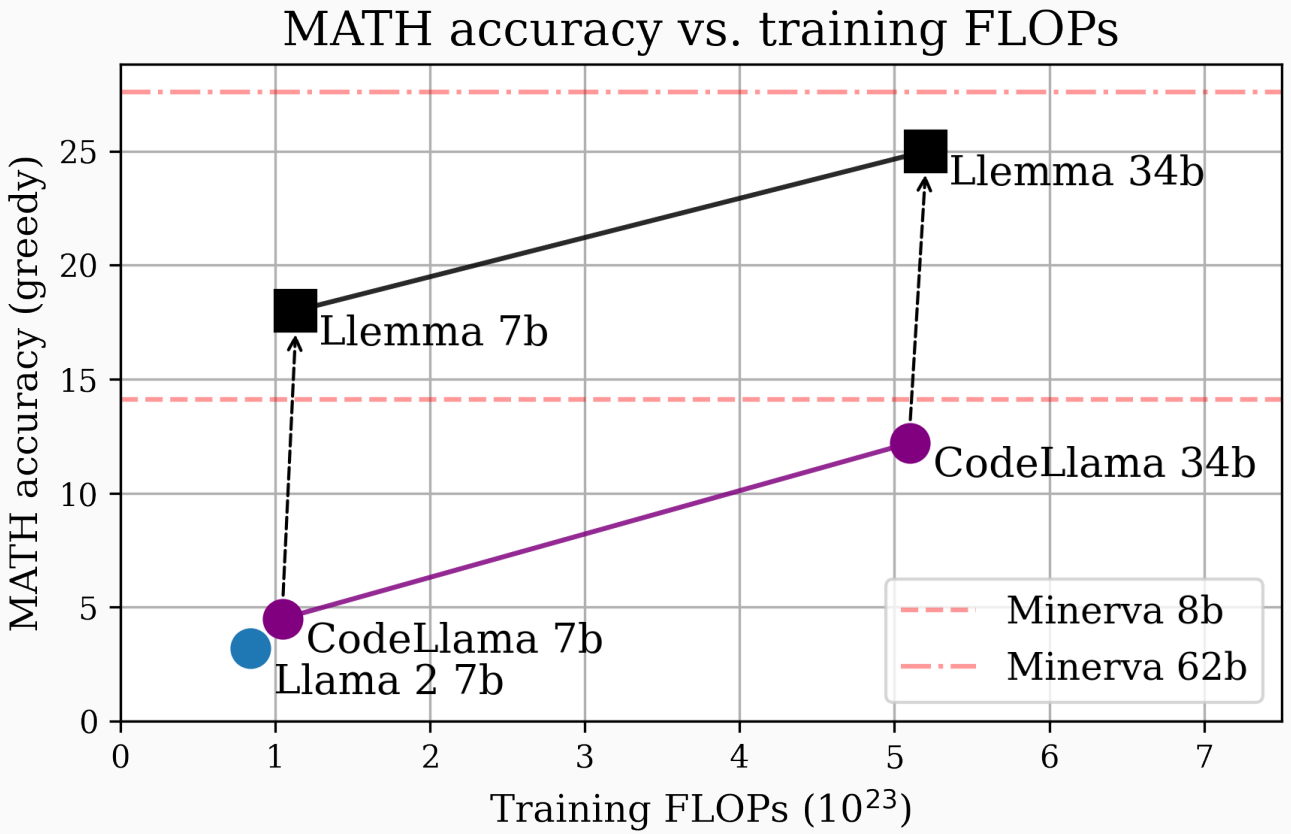


Figure 5: Llemma improves with a modest amount of math-specific compute

Data: Proofpile II

Proofpile II: 55 billion tokens[3]

- Code: 11B tokens
- Web: 15B tokens
- Papers: 29B tokens

[3] Available at huggingface.co/datasets/EleutherAI/proof-pile-2

Code: AlgebraicStack

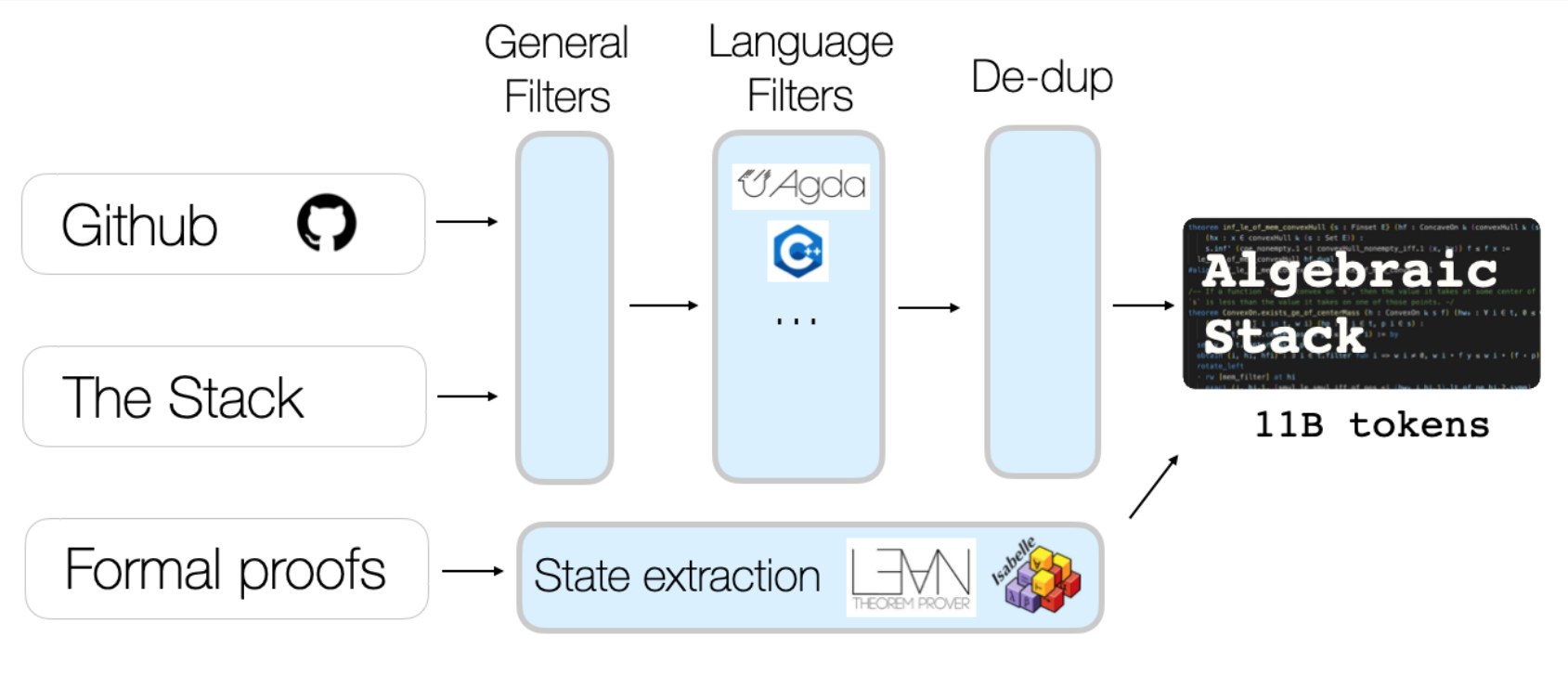


Figure 6: AlgebraicStack pipeline

Web: OpenWebMath

Web: OpenWebMath [Paster et al 2023][4]

- 14.7 billion tokens of math-related web data

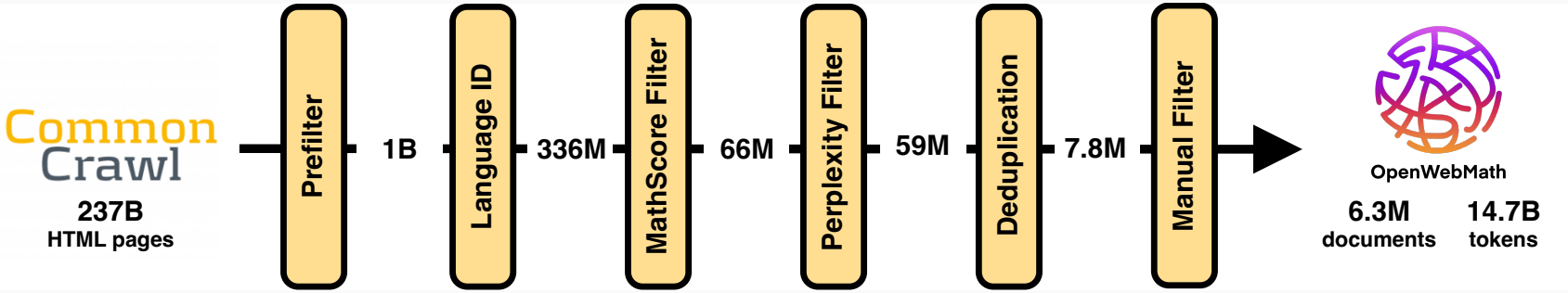


Figure 7: OpenWebMath pipeline.

[4] *OpenWebMath: An Open Dataset of High-Quality Mathematical Web Text.* Keiran Paster, Marco Dos Santos, Zhangir Azerbayev, Jimmy Ba

Problem solving with chain-of-thought

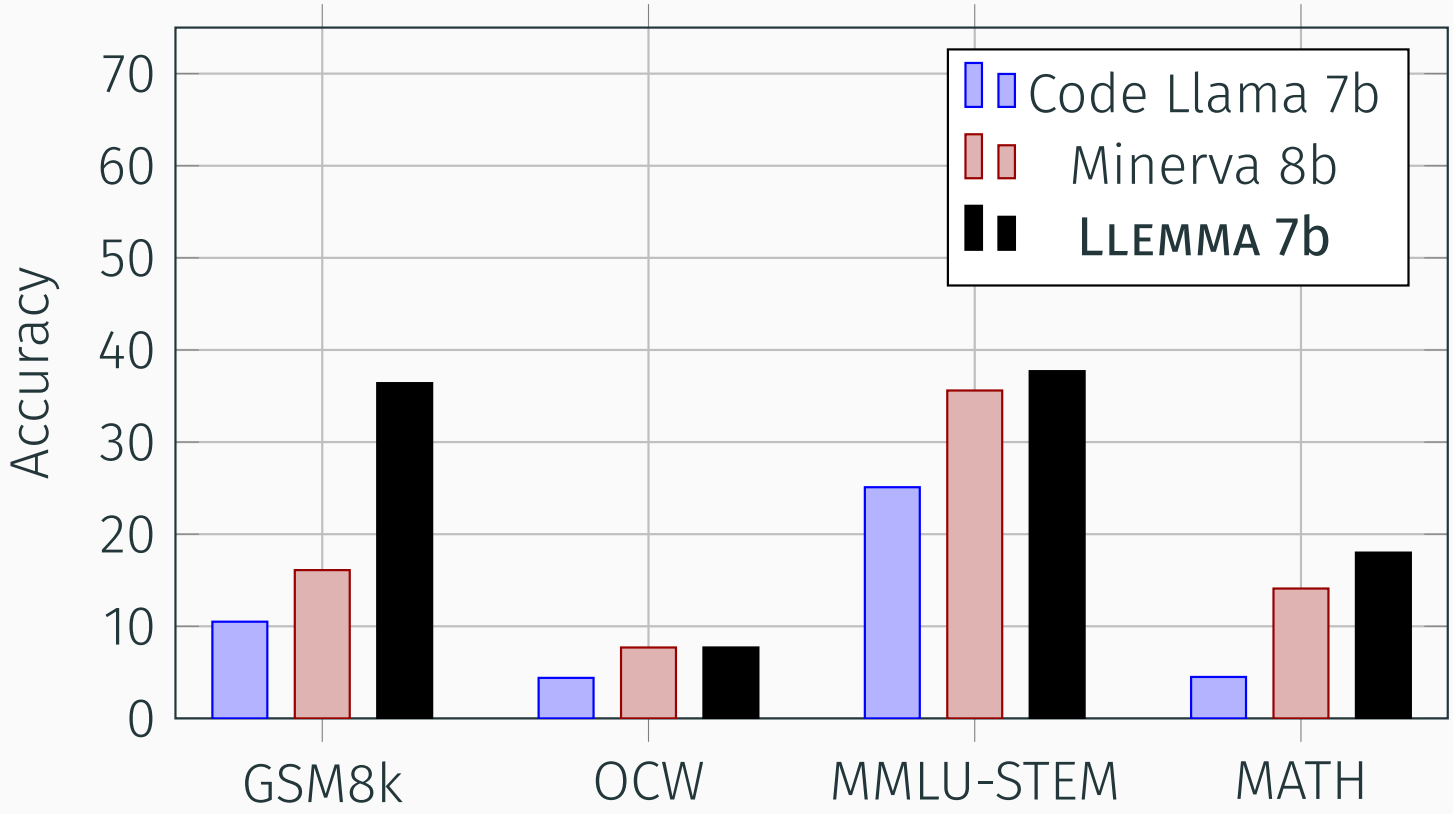


Figure 8: Few-shot problem solving (greedy decoding)

LLEMMA formal-to-formal theorem proving

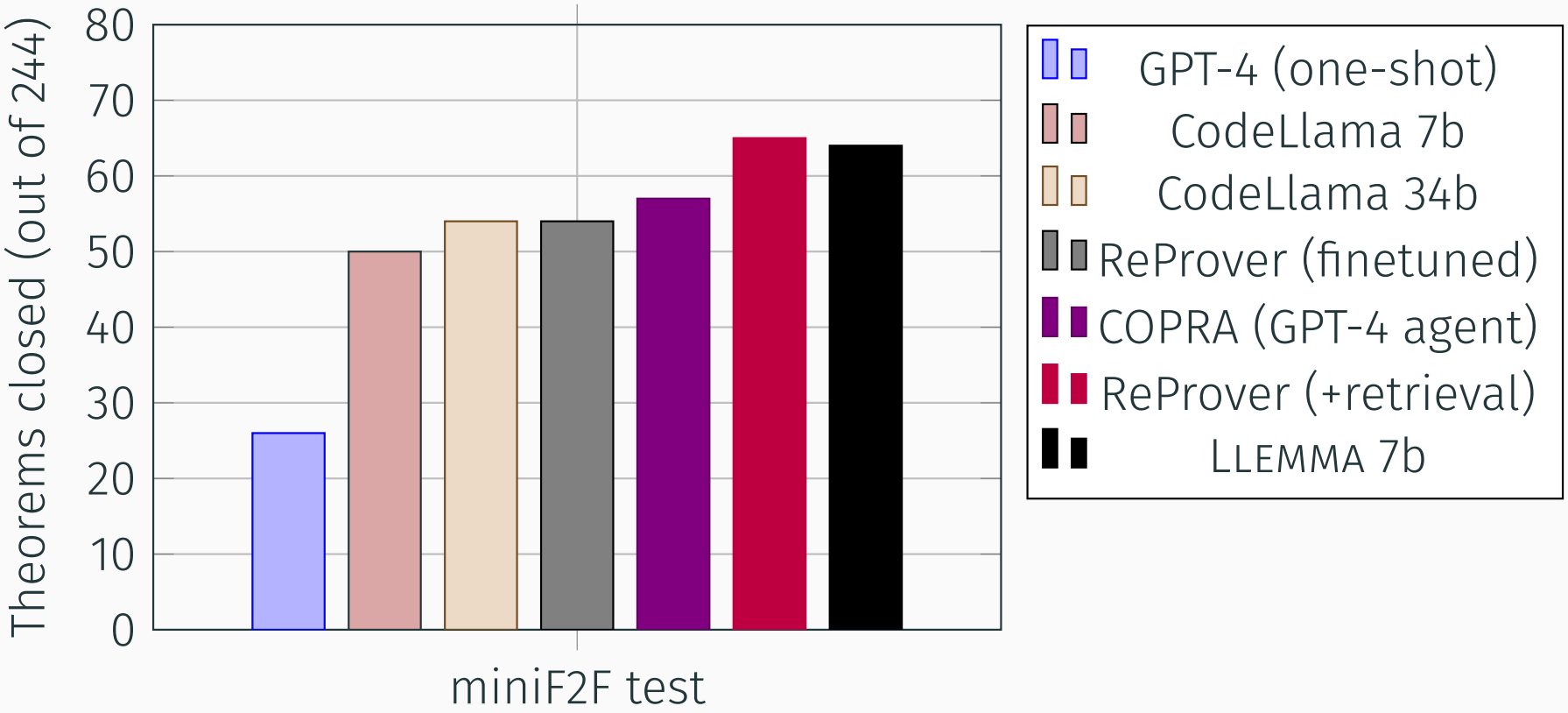


Figure 9: Few-shot formal-2-formal proving with LLEMMA

Integrating LLEMMA and the Lean theorem prover

LLMLEAN: tools that integrate LLMs into the Lean proof assistant

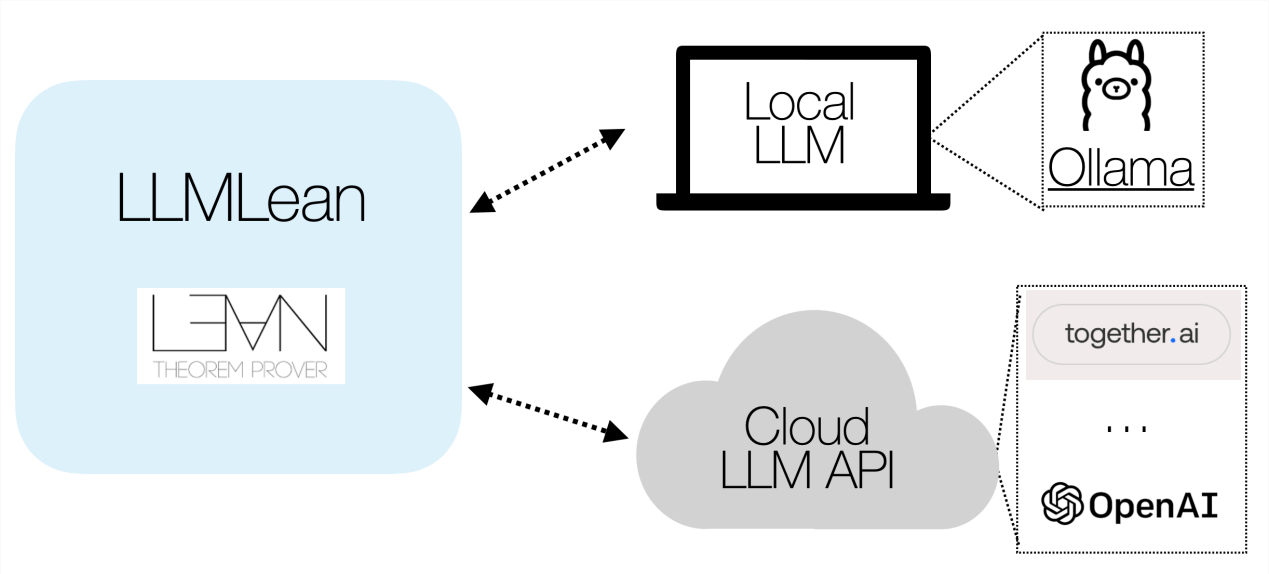


Figure 10: github.com/wellecks/llmlean

Based on *llmstep: LLM proofstep suggestions in Lean* [Welleck & Saha 2023]

Integrating LLEMMA and the Lean theorem prover

Example: Verified LLEMMA suggestions on a MacBook Pro:

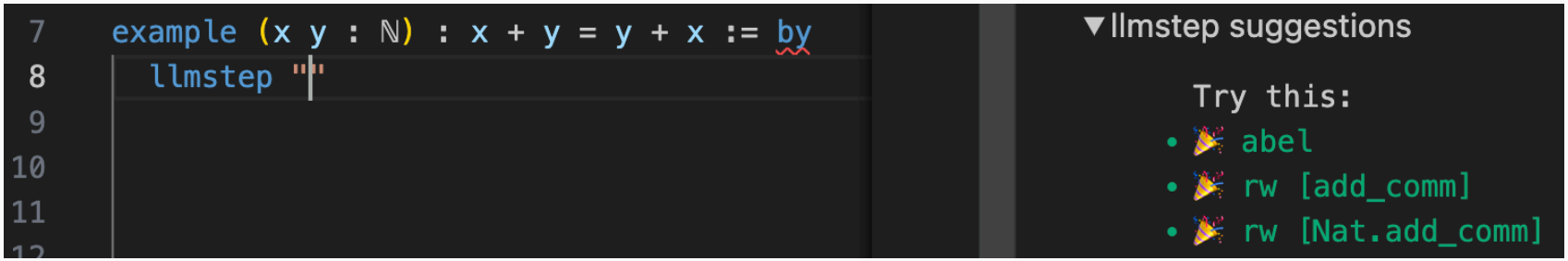


Figure 11: github.com/wellecks/llmlean

Based on *llmstep: LLM proofstep suggestions in Lean* [Welleck & Saha 2023]

Recap

- Recipe for specializing a language model to mathematics
  - Llemma: 7B and 34B CodeLLama further trained on Proofpile II
- Open platform for research:
  - Code: *https://github.com/EleutherAI/math-lm*
  - Models: *https://huggingface.co/EleutherAI/llemma_7b*
  - Data: *https://huggingface.co/datasets/EleutherAI/proof-pile-2*

Recap

- Focus on data *quality* and *transfer* can enable efficient scaling

Recap

- Focus on data *quality* and *transfer* can enable efficient scaling

Next: going *beyond* human data

II. Easy-to-hard generalization

II. Easy-to-hard generalization

Easy-to-Hard Generalization: Scalable Alignment Beyond Human Supervision

Zhiqing Sun*, Longhui Yu*, Yikang Shen, Weiyang Liu
Yiming Yang^{+}, Sean Welleck^{+}, Chuang Gan^{+}

arXiv 2024.

Easy-to-hard generalization

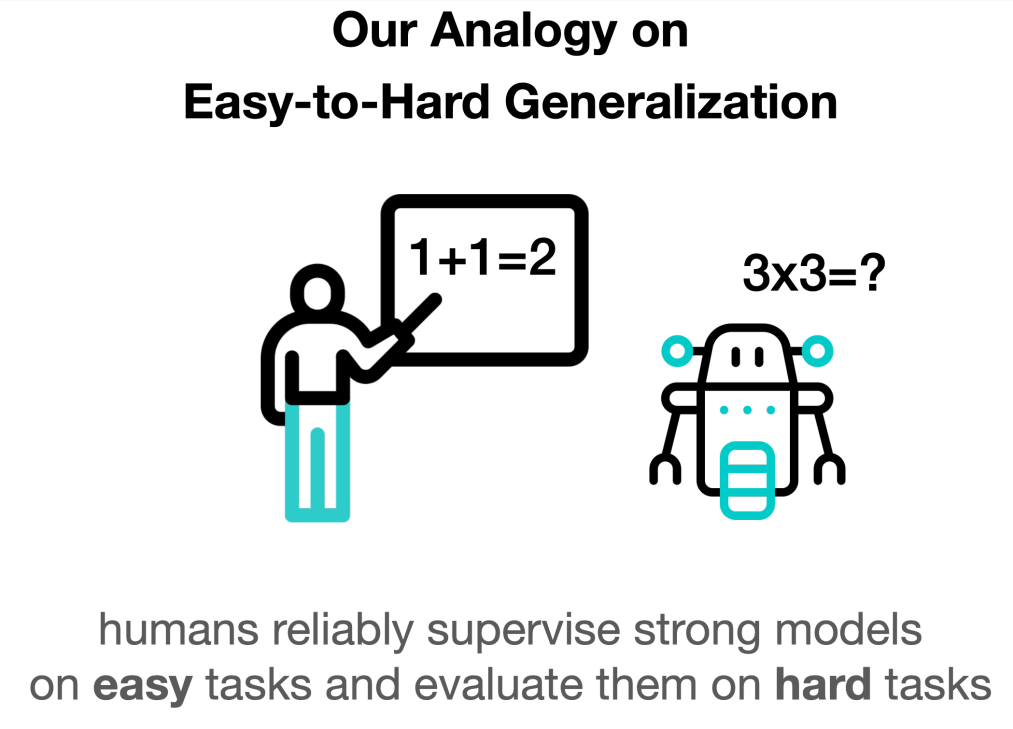


Figure 12: Easy-to-hard generalization

Easy-to-hard generalization

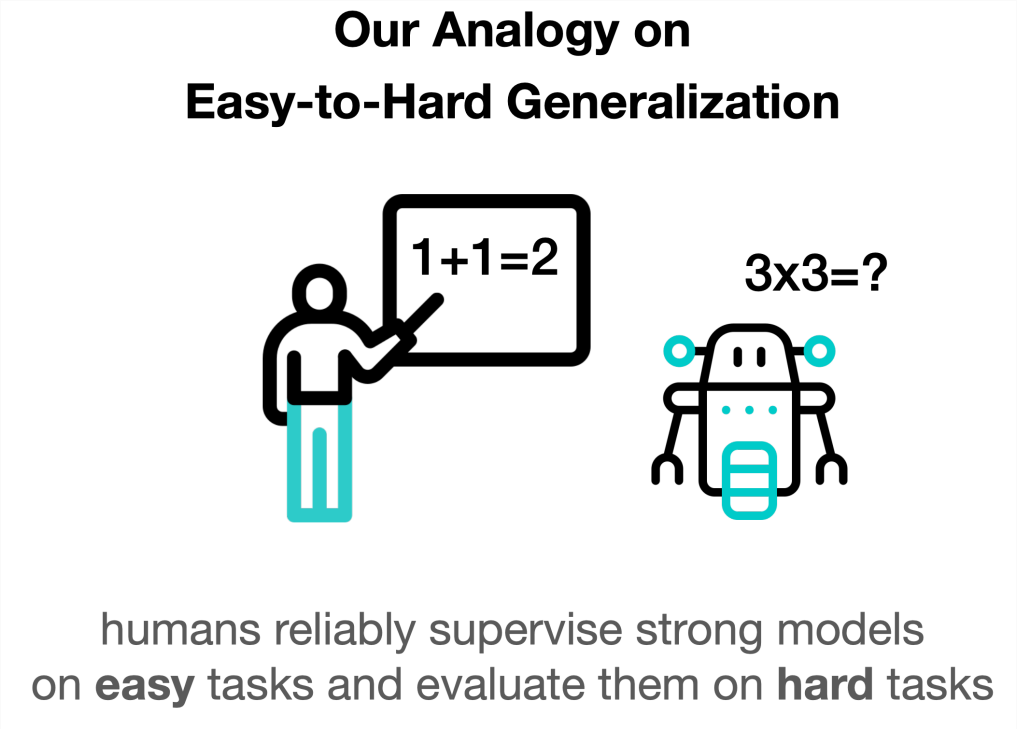


Figure 12: Easy-to-hard generalization

Operationalize as:

- **Train** on **easy** problems: MATH level 1-3
- **Test** on **hard** problems: MATH level 4-5

Easy-to-hard generalization

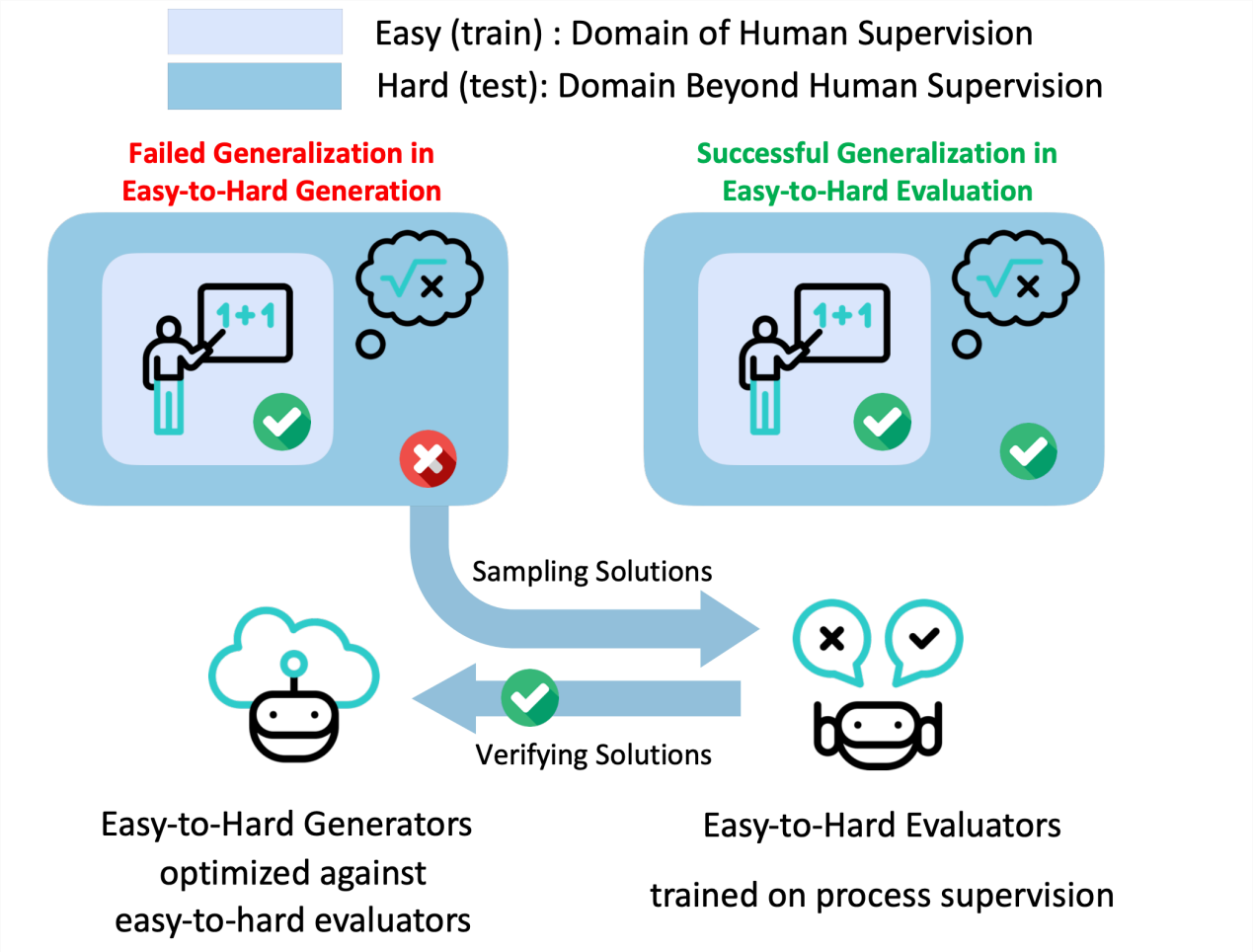


Figure 13: *Key idea:* easy-to-hard evaluation can facilitate easy-to-hard generation

Easy-to-hard generalization

- Train an evaluator $g_\phi(y) \to [0, 1]$ on easy problems
- Use evaluator to select generated solutions $\{y^n\}_{n=1}^N$ on hard problems

Evaluator: *OPRM*

Evaluator: Outcome-process reward model (OPRM)

- Outcome: given solution y, predict correctness[5]
- Process: given steps $y_1, \ldots, y_k$, predict correctness[6]

OPRM is trained to predict both

[5] *E.g., Cobbe et al 2021, Training Verifiers to Solve Math Word Problems*
[6] *E.g., Lightman et al 2023, Let's Verify Step-by-Step*

Inference-time

Select a solution by **weighted majority voting**:

- Generate many solutions (e.g. 1024)
- Score each solution using the evaluator $g_\phi(y)$
- Group the solutions by answer, choose group with highest score

Inference-time scaling on *hard* problems

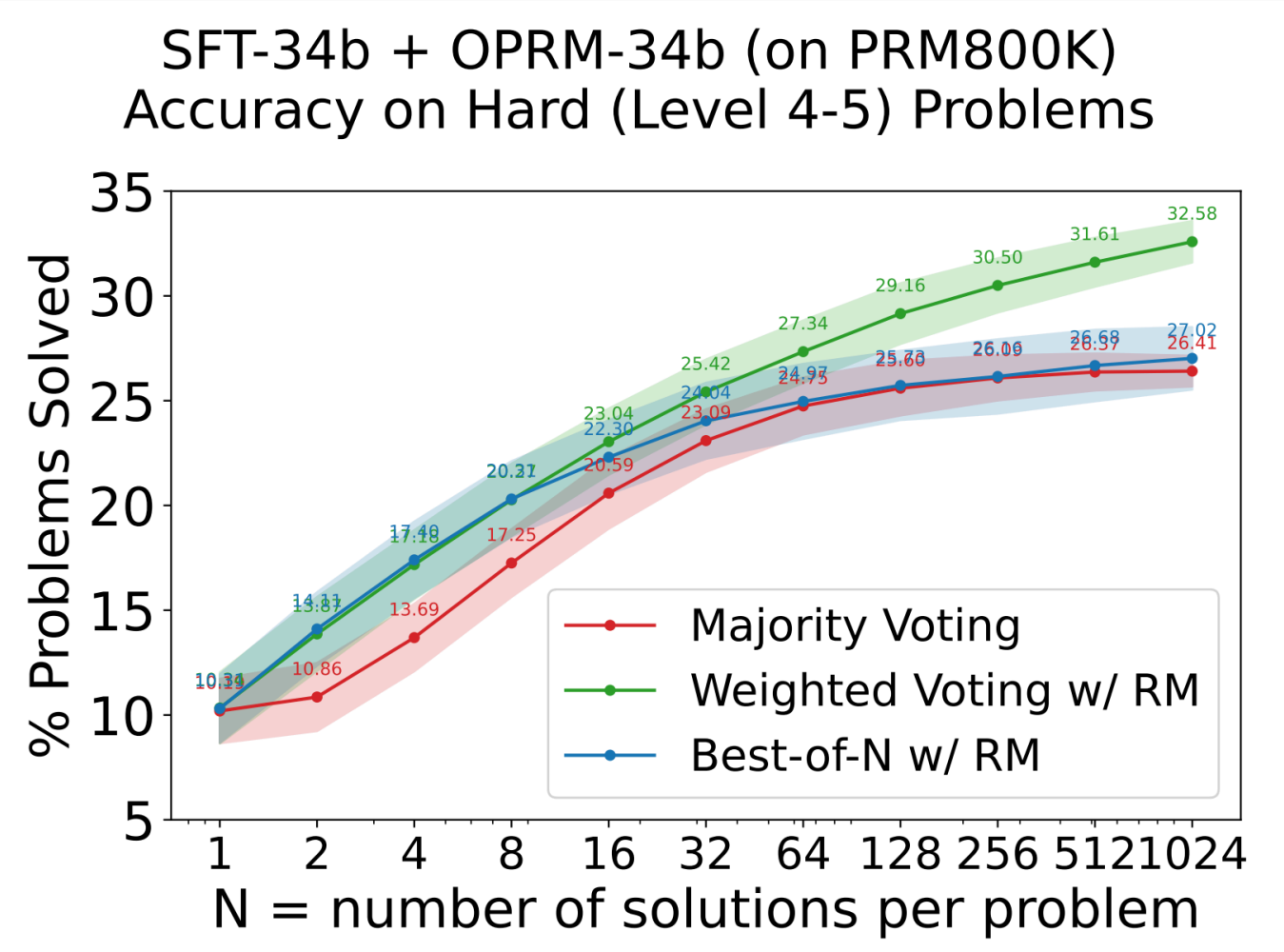


Figure 14: OPRM inference scaling on hard problems

Inference-time scaling on *all* problems

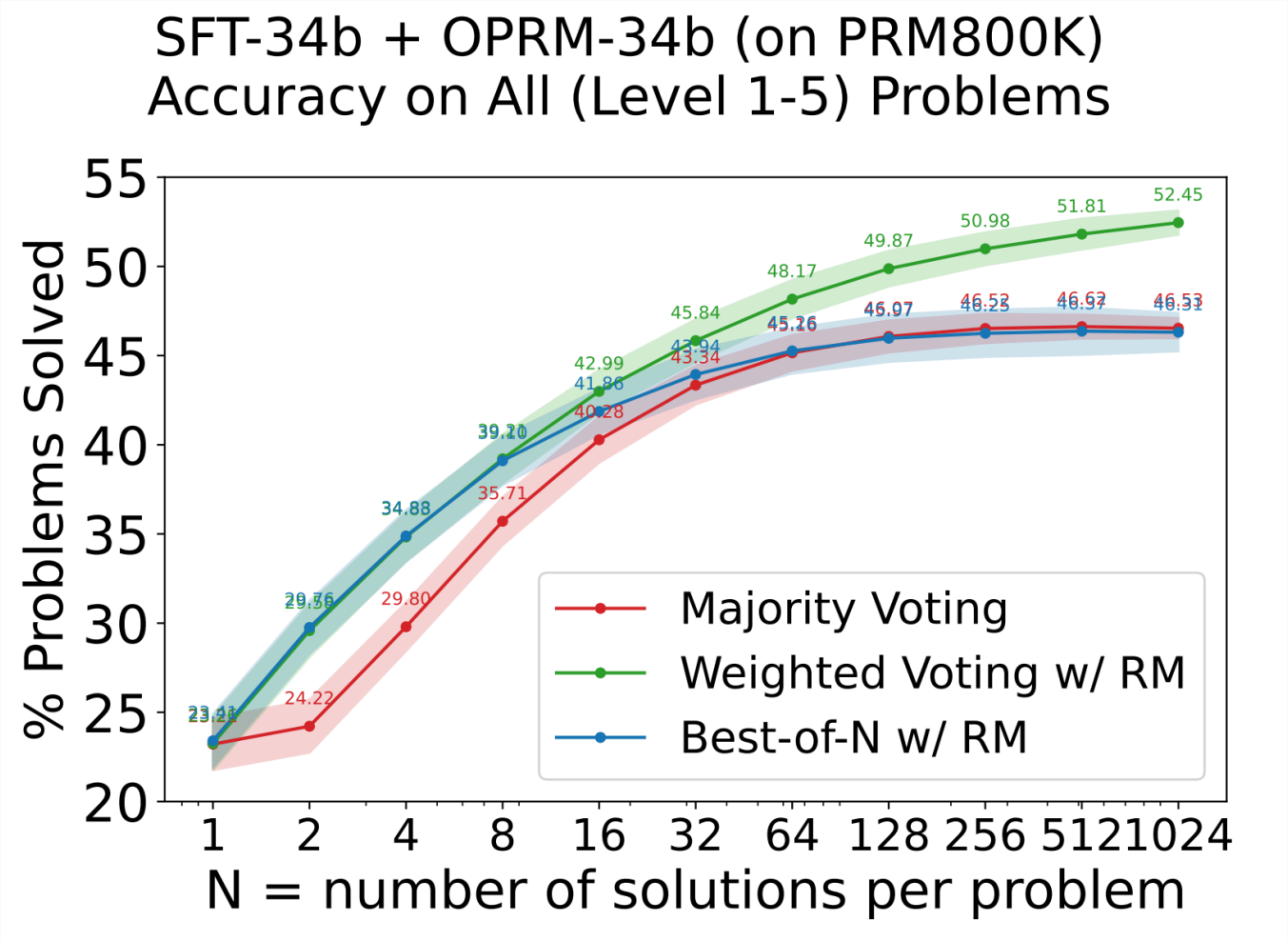
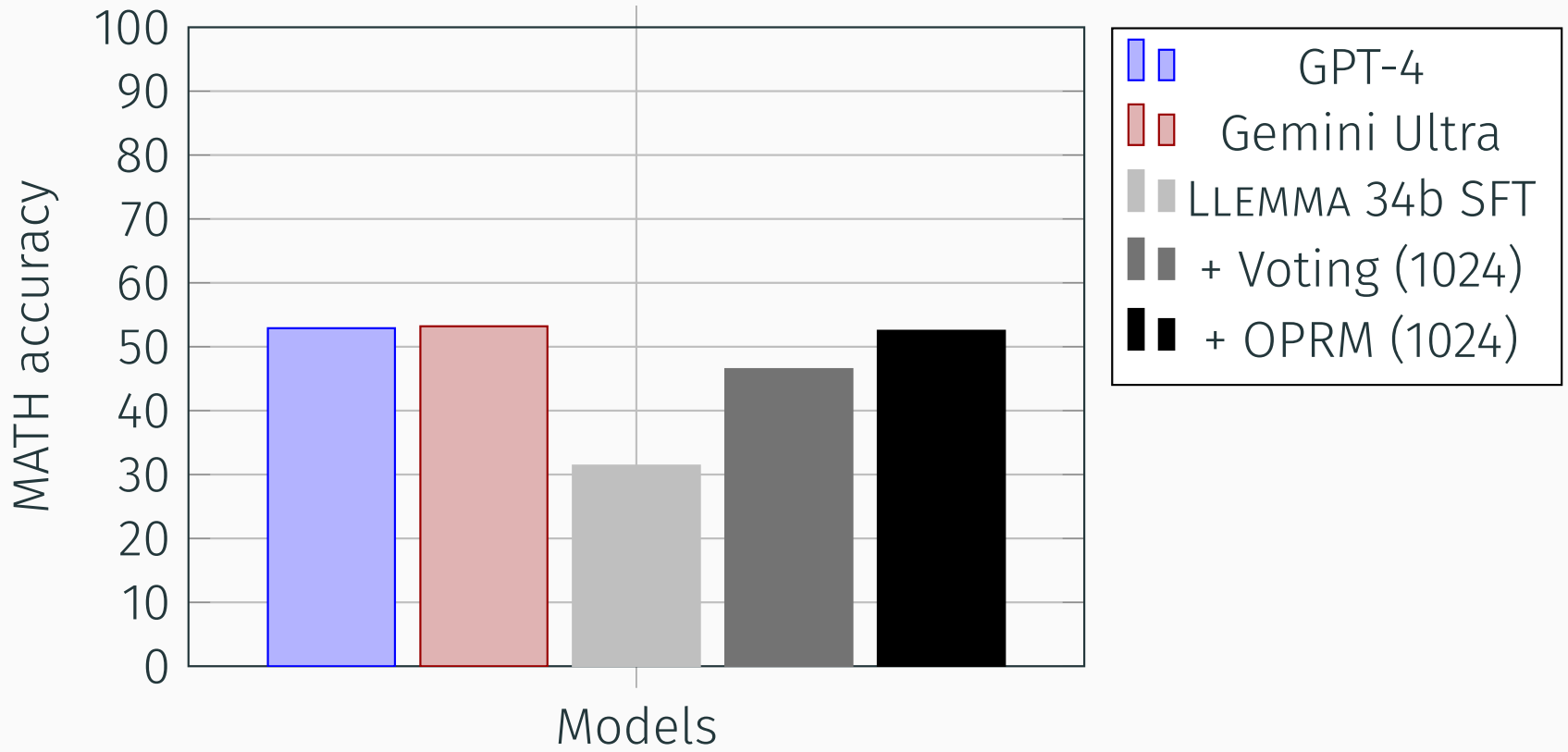


Figure 15: OPRM inference scaling on hard problems

Performance (all problems)

Comparison with black-box API models (GPT-4, Gemini Ultra):[7]



[7] Experiment setting: *PRM 800k*, 34B model

Using the evaluator for reinforcement learning

1. Generate solutions on easy and hard problems
2. Use easy-to-hard evaluator as a reward function

Using the evaluator for reinforcement learning

Outperforms finetuning on *all* problems:[8]

[8] Experiment setting: 7B model, RL with PPO

Recap

1. Scaling data *quality* and *quantity*
    - Llemma and ProofPile II: high-quality data and transfer
2. Scaling *inference* and *evaluation*
    - Scalable evaluation can facilitate easy-to-hard generalization

Looking ahead | data scaling

New data algorithms for mining high-quality data

- E.g., DeepSeek-Math [5]:

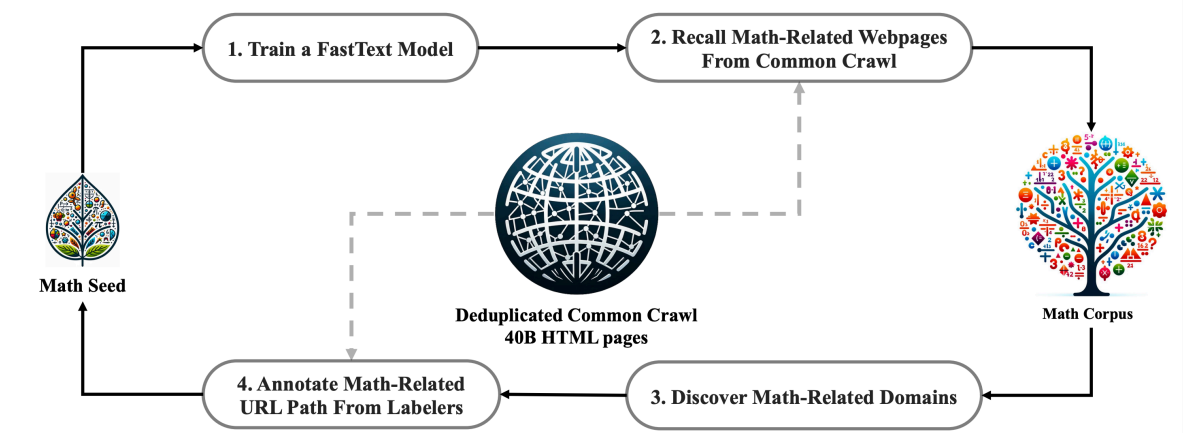


Figure 2 | An iterative pipeline that collects mathematical web pages from Common Crawl.

And synthetic data…

Looking ahead | scaling evaluation

Scalable evaluation beyond 0/1 correctness

- E.g. language feedback and fine-grained rubrics [Kim et al 2024 [3]]

Looking ahead

How do we enable systems that discover new knowledge and improve over time?

Thank you!

- Zhangir Azerbayev (Princeton)
- Hailey Schoelkopf (Eleuther)
- Keiran Paster (Toronto, Vector)
- Marco Dos Santos (Cambridge)
- Stephen McAleer (CMU)
- Albert Jiang (Cambridge)
- Jia Deng (Princeton)
- Stella Biderman (Eleuther)
- Zhiqing Sun (CMU)
- Longhui Yu (Peking U)
- Yikang Shen (MIT-IBM Watson AI Lab)
- Weiyang Liu (Cambridge, MPI)
- Yiming Yang (CMU)
- Chuang Gan (MIT-IBM Watson AI Lab, UMass Amherst)

https://github.com/EleutherAI/math-lm

https://github.com/Edward-Sun/easy-to-hard

Sean Welleck (CMU)
Learning, Language, and Logic (L3) Lab

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